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Propagation of Sound Waves in a Moving Medium

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SUMMARY

The propagation of sound waves radiated by a two-dimensional source in a fluid moving with subsonic velocity between two perfectly reflecting parallel walls is considered. The steady state problem appears to have a non-unique solution, for which Sommerfeld's radiation condition does not apply. Two methods are used for obtaining the unique solution. First the corresponding problem for a fluid with non-zero bulk viscosity is solved, which has a unique solution and then the limit for zero bulk viscosity is taken. Secondly, the initial value problem for a source being switched on at time $t=0$ is solved and it is shown that its solution tends to the same steady state solution in the limit for $t\to\infty$. In the last section the results for the corresponding axisymmetric case are given. In the appendix some properties of the two-dimensional steady state solution are explained qualitatively.

1. Introduction

In the linear theory of propagation of waves such as acoustic, electromagnetic or gravity waves, one is usually interested in the situation occurring after an infinite period of time. Particularly, in the case of a homogeneous medium, being undisturbed at time $t=0$, with a source (or a number of sources) being switched on at $t=0^+$, emitting harmonic waves with constant frequency ω and constant amplitude, it is assumed generally that for $t\to\infty$ the functions describing the wave phenomenon, such as a velocity potential, an electromagnetic potential, or a surface elevation, all become harmonically time-dependent with the same frequency ω (monochromatic waves). Mathematically this means that an initial-boundary-value problem is replaced by a pure boundary value problem for $t\to\infty$. The initial value problem, if properly posed, has a unique solution, also for t tending to infinity, but the assumed limit problem for $t\to\infty$, which will be called the steady state problem in the sequel, does not have a unique solution in general. In many cases, however, the wellknown Sommerfeld radiation condition may be used to determine a unique solution to the steady state problem. However, this radiation condition which, roughly speaking, forbids incoming waves generated at infinity, may only be used for an unbounded homogeneous medium with all disturbances, such as sources and boundaries not extending to infinity.

In this article a problem is treated for which the Sommerfeld radiation condition does not apply and where the non-uniqueness of the steady state solution should be relieved by other means. We will consider a two-dimensional acoustic source, situated between two perfectly reflecting parallel walls, both extending to infinity in both directions. Between these walls flows an inviscid fluid with constant subsonic velocity parallel to the walls. At time $t=0$ the source is switched on and emits a constant harmonic sound wave for $t>0$. The steady state problem will appear to possess a non-unique solution, and because the boundaries extend to infinity, no use can be made of the Sommerfeld radiation condition. We will use two methods for obtaining the unique solution. In section 4 a small artificial bulk viscosity is introduced in the fluid which yields a unique steady state solution. By allowing this bulk viscosity to approach zero, the unique steady state solution to our problem with zero bulk viscosity is obtained.

A second and mathematically more convincing method is to solve the complete initial value problem and to obtain the solution in the limit for $t\to\infty$. This is done in section 5 and it is analogous to a method used by Stoker [1,2] for an initial value problem for gravity waves. In this way we will show that the same steady state solution appears as obtained by the bulk viscosity method.

Furthermore we will show that the steady state solution to our problem exhibits a number of unexpected properties. For instance it will appear that under special circumstances undamped incoming waves, generated at infinity are possible as parts of our solution. This phenomenon, which has been described first by Le Grand [3], will be explained qualitatively in the Appendix.

In section 6 the analogous axisymmetric problem of a point source radiating sound waves in a circular pipe is treated. The results for the initial value problem are given and in a way parallel to the two-dimensional case it is shown briefly that a unique steady state solution appears in the limit $t \to \infty$.

Throughout this paper all functions are assumed to be generalised functions in the sense of Lighthill [4]. This allows us to make use of δ-functions and to take Fourier transforms without restrictions on boundedness at infinity or integrability. The results obtained in this way should be interpreted as generalised functions, but in our problems this will cause no difficulties because the results are readily seen to be interpretable as functions in the ordinary sense in any region that does not contain the origin.

2. Formulation of the problem

Consider in a three-dimensional $\bar{x}$, $\bar{y}$, $\bar{z}$-space a region bounded by two parallel walls given by $\bar{y} = \pm d$, $-\infty < \bar{x} < \infty$ and $-\infty < \bar{z} < \infty$. This region is filled with a non-viscous compressible fluid moving with constant velocity U in the positive $\bar{x}$-direction. A harmonically fluctuating acoustic line source is situated on the z-axis, extending to infinity in both directions. The problem may be treated therefore as a two-dimensional problem, i.e. independent of the $\bar{z}$-coordinate. The source is switched on at time $\bar{t} = 0$. We assume the fluid velocity to be subsonic, i.e. $U < c$, where c denotes the constant sound velocity.

In a coordinate system (x_m, y_m, t_m) moving with the undisturbed fluid the velocity potential $\bar{\psi}(x_m, y_m, t_m)$ of the sound waves satisfies the inhomogeneous wave equation (c.f. for example Friedlander [5], Landau and Lifshits [6]):

$$\bar{\psi}_{x_m x_m} + \bar{\psi}_{y_m y_m} - \frac{1}{c^2}\bar{\psi}_{t_m t_m} = \delta(x_m + Ut_m)\delta/y_m)e^{i\omega t_m}, \quad t_m > 0 \tag{2.1}$$

where the righthand member accounts for the harmonic source of unit strength and frequency ω moving with velocity U in the direction of the negative x_m-axis. Returning to the original coordinates $\bar{x}$, $\bar{y}$ and $\bar{t}$ by means of $\bar{x} = x_m + Ut_m$, $\bar{y} = y_m$, $\bar{t} = t_m$, equation (2.1) becomes:

$$(1 - M^2)\bar{\psi}_{\bar{x}\bar{x}} + \bar{\psi}_{\bar{y}\bar{y}} - \frac{2M}{c}\bar{\psi}_{\bar{x}\bar{t}} - \frac{1}{c^2}\bar{\psi}_{\bar{t}\bar{t}} = \delta(\bar{x})\delta(\bar{y})e^{i\omega\bar{t}}, \quad \bar{t} > 0 \tag{2.2}$$

where $M = U/c$ denotes the Machnumber of the flow ($M < 1$). Introducing the dimensionless quantities

$$x = \bar{x}d^{-1}, \quad y = \bar{y}d^{-1}, \quad t = \omega\bar{t}, \quad \psi = \bar{\psi}\omega^{-1}d^{-2}, \quad k = \bar{k}d = \omega d/c\,,$$

we get the following initial-boundary-value problem for $\psi(x, y, t)$:

$$(1 - M^2)\psi_{xx} + \psi_{yy} - 2kM\psi_{xt} - k^2\psi_{tt} = \delta(x)\delta(y)e^{it}, \quad t > 0 \tag{2.3}$$

with initial conditions for $t = 0$:

$$\psi(x, y, 0) = \psi_t(x, y, 0) = 0\,. \tag{2.4}$$

Assuming perfectly reflecting walls the boundary conditions become for $y = \pm 1$:

$$\psi_y(x, 1, t) = \psi_y(x, -1, t) = 0 \tag{2.5}$$

3. The steady state problem

Formulating the steady state problem it is assumed that for $t\to\infty$ the potential $\psi(x, y, t)$ depends harmonically on t with a dimensionless frequency equal to the frequency of the waves emitted by the source:

$$\psi(x, y, t)=\psi^*(x, y)e^{it}.$$

In fact, our initial value problem may be replaced by the equivalent problem of an infinite array of mirror sources, each of which emits waves in a moving medium without boundaries ([7]). Because each of these single sources, being switched on at $t=0$, will emit sound waves of which the potential for $t\to\infty$ depends harmonically on time, we may expect the steady state solution to our problem to exist. (c.f. Appendix).

The steady state boundary value problem for $\psi^*(x, y)$ becomes after omission of the asterisk:

$$\beta^2\psi_{xx}+\psi_{yy}-2ikM\psi_x+k^2\psi=\delta(x)\delta(y), \qquad \beta^2=1-M^2, \tag{3.1}$$

with boundary conditions

$$\psi_y(x, 1)=\psi_y(x, -1)=0 . \tag{3.2}$$

Solving this boundary value problem by the Fourier transform technique, we introduce:

$$\Psi(\lambda, y)=\int_{-\infty}^{\infty} e^{i\lambda x}\psi(x, y)dx . \tag{3.3}$$

For this transformed potential $\Psi(\lambda, y)$ we have the ordinary differential equation

$$\Psi_{yy}-\gamma^2(\lambda)\Psi=\delta(y), \quad \gamma(\lambda)=(\beta^2\lambda^2+2kM\lambda-k^2)^{\frac{1}{2}} \tag{3.4}$$

with boundary conditions

$$\Psi_y(\lambda, 1)=\Psi_y(\lambda, -1)=0 . \tag{3.5}$$

The function $\gamma(\lambda)$ is made a single-valued function of λ by cutting the complex λ-plane along the part of the real λ-axis between the two branch points $\lambda=k(1+M)^{-1}$ and $\lambda=-k(1-M)^{-1}$ and by choosing that branch of $\gamma(\lambda)$ which behaves like $\beta\lambda$ for $|\lambda|\to\infty$. The solution of equation (3.4) satisfying boundary conditions (3.5) is

$$\Psi(\lambda, y)=\frac{1}{2\gamma}\,\frac{e^{\gamma(|y|-1)}+e^{-\gamma(|y|-1)}}{e^{-\gamma}-e^{\gamma}} . \tag{3.6}$$

The formal solution of our original problem follows by inversion of $\Psi(\lambda, y)$:

$$\psi(x, y)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\Psi(\lambda, y)d\lambda . \tag{3.7}$$

The integrand in equation (3.7) is an analytic function of λ apart from an infinite number of simple poles in the λ-plane, a finite number of which lies on the real λ-axis. The poles $\lambda=\xi_n^{\pm}$ follow from the equation:

$$\gamma(\lambda)=\pm n\pi i, \qquad n=0, 1, 2, \ldots \tag{3.8}$$

yielding after some calculation:

$$\xi_n^{\pm}=-\beta^{-2}\big(kM\pm\sqrt{k^2-\beta^2n^2\pi^2}\big), \qquad n\leqq N,$$

$$\xi_n^{\pm}=-\beta^{-2}\big(kM\mp i\sqrt{\beta^2n^2\pi^2-k^2}\big), \qquad n>N$$

where N is the largest natural number $\leqq k\pi^{-1}\beta^{-1}$.

The $2N+2$ poles $\xi_0^{\pm}, \xi_1^{\pm}, \ldots \xi_N^{\pm}$ on the real λ-axis give rise to difficulties in the evaluation of integral (3.7), because the path of integration is running through these poles. In this case we

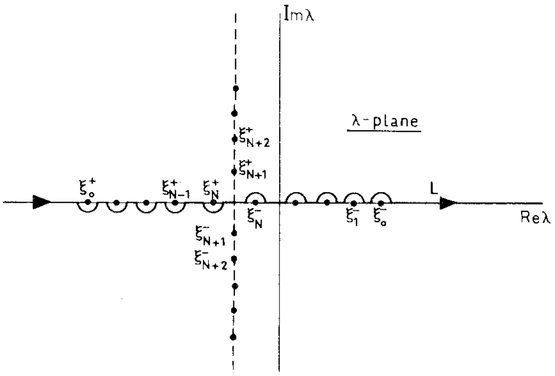

Figure 1. The poles of $\Psi(\lambda, y)$ and the path of integration L.

cannot use the Sommerfeld radiation condition in order to decide to which halfplane each of these poles belong.

In the next section, therefore, the analogous problem for a fluid with non-zero bulk viscosity is solved. In that case the poles do not lie on the contour of integration. Taking the limit for bulk viscosity zero, we are able to decide about the position of the poles with respect to the contour of integration. This method has been used by Le Grand [3] for similar problems.

4. The steady state solution as the limit case for zero bulk viscosity

Introducing a small non-zero bulk viscosity coefficient $\bar{l}$ we take into account a damping caused by the frictional effect in the fluid due to pure compression and expansion. The basic wave equation corresponding to equation (2.1) then becomes, ([3,6]):

$$\left(\frac{\partial^2}{\partial x_m^2}+\frac{\partial^2}{\partial y_m^2}\right)\left(\bar{\psi}+\frac{\bar{l}}{c}\frac{\partial\bar{\psi}}{\partial t_m}\right)-\frac{1}{c^2}\frac{\partial^2\bar{\psi}}{\partial t_m^2}=\delta(x_m+Ut_m)\delta(y_m)e^{i\omega t_m}, \quad t_m>0$$

and the steady state boundary value problem analogous to equations (3.1) and (3.2) becomes with $l=\bar{l}d$:

$$\beta^2\psi_{xx}+\psi_{yy}+\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)l(M\psi_x+ik\psi)-2ikM\psi_x+k^2\psi=\delta(x)\delta(y)\,. \tag{4.1}$$

Taking Fourier transforms with respect to x we get the same differential equation (3.4) with solution (3.6) for $\Psi(\lambda, y)$, but with $\gamma^2(\lambda)$ replaced by

$$\gamma_l^2(\lambda)=\frac{(1-M^2)\lambda^2+2kM\lambda-k^2+l^2(\lambda M-k)-il(\lambda M-k)^2}{1+l^2(\lambda M-k)^2}, \tag{4.2}$$

which may be written for small values of l as:

$$\gamma_l^2(\lambda)=(1-M^2)\lambda^2+2\{k-\tfrac{1}{2}il(M\lambda-k)^2\}M\lambda-\{k-\tfrac{1}{2}il(M\lambda-k)^2\}^2+O(l^2)\,. \tag{4.3}$$

This equation shows after comparison with the corresponding one for zero bulk viscosity, that the introduction of a small bulk viscosity gives a slight shift in k with negative imaginary part, causing an exponential decay in our wave solution.

For small non-zero bulk viscosity, assume that the poles of in the complex λ-plane are analytic functions of l in a small neighbourhood of $l=0$ and hence may be expanded as follows

$$\xi_n(l)=\xi_n^{\pm}+l\eta_n+O(l^2)\,, \qquad n=0, 1, 2, \ldots \tag{4.4}$$

where $\xi_n^{\pm}$ denote the poles in the case of zero bulk viscosity. Substituting eq. (4.4) in the equation $\gamma_l^2(\lambda)+n^2\pi^2=0$, $n=0, 1, 2, \ldots$, and linearizing with respect to l we find:

$$(1-M^2)[\xi_n^\pm+2l\eta_n\xi_n^\pm]+2kM[\xi_n^\pm+l\eta_n]-k^2-il[\xi_n^\pm M-k]^3=-n^2\pi^2\,,$$
$$n=0,1,2,\ldots$$

Because of the fact that the $\xi_n^\pm$ satisfy eq. (3.8), we find approximatively for the shifts η_n of the poles $\xi_n^\pm$:

$$\eta_n=\frac{\frac{1}{2}i(M\xi_n^\pm-k)^3}{(1-M^2)\xi_n^\pm+kM}\,,\qquad n=0,1,2,\ldots \tag{4.5}$$

Now it is easily seen that the shift is negative imaginary for the poles ξ_n^-, $n=0, 1, \ldots, N$, situated on the real λ-axis to the right of the axis of symmetry Re $\lambda=-kM\beta^{-2}$. Thus in the case of non-zero bulk viscosity these poles belong to the lower halfplane. Analogous the poles ξ_n^+, $n=0, 1, \ldots, N$, get a positive imaginary shift and hence belong to the upper halfplane. In the limit case of zero bulk viscosity the contour integration should be taken as indicated in fig. 1.

By the residue theorem the solution of our steady state problem with zero bulk viscosity is found for $x>0$ as the sum of the residues of the poles ξ_n^-, $n=0, 1, 2, \ldots$, and for $x<0$ as the sum of the residues of the poles ξ_n^+, $n=0, 1, 2, \ldots$, by closing the contour with large semi-circles in the lower resp. upper halfplane, the contributions of which vanish for the radius tending to infinity (c.f. Doetsch [8]):

$$\psi(x,y,t)=-i\sum_{j=0}^{\infty}\lim_{\lambda\to\xi_j^-}\frac{\lambda-\xi_j^-}{2\gamma(e^{-\gamma}-e^{\gamma})}\cos\{j\pi(|y|-1)\}e^{i(t-\xi_j^-x)},\quad x>0 \tag{4.6a}$$

$$\psi(x,y,t)=-i\sum_{j=0}^{\infty}\lim_{\lambda\to\xi_j^+}\frac{\lambda-\xi_j^+}{2\gamma(e^{-\gamma}-e^{\gamma})}\cos\{j\pi(|y|-1)\}e^{i(t-\xi_j^+x)},\quad x<0 \tag{4.6b}$$

This steady state solution has a number of unexpected properties which will be examined by considering the physical meaning of the individual residues. Consider for the moment the region $x>0$, with $\psi(x, y, t)$ given by (4.6a). Each residue represents a travelling wave: for Re $\xi_n^->0$ it is a wave moving to the right (outgoing wave) and for Re $\xi_n^-<0$ it is a wave moving to the left (incoming wave). For each value of the Mach-number $M>0$ there is an infinite number of waves coming in from infinity and only a finite number of outgoing waves, corresponding to the finite number of poles ξ_i^-, $i=0, 1, \ldots$ $m=[k\pi^{-1}]$, on the positive part of the real λ-axis. Note that this number m is independent of M.

The majority of the incoming waves has an amplitude tending to zero for $x\to\infty$, namely all waves corresponding to the poles ξ_n^-, $n=N+1, N+2, \ldots$, with $N=[k\pi\beta^{-1}]$. This is due to the negative imaginary part of these poles, resulting in an exponential decay in x. Depending on M and k there are one or more poles $\xi_{m+1}^-, \ldots\ \xi_{N-1}^-, \xi_N^-$ on the negative part of the real axis, namely if $kM-(k^2-\beta^2N^2\pi^2)^{\frac{1}{2}}>0$. These poles yield undamped incoming waves generated at infinity! However, for $x<0$ we only get diverging waves. Note that for $M=0$ the axis of symmetry of the poles coincides with the imaginary λ-axis and hence in that case no waves incoming from infinity will occur. Thus for the case $M=0$ a radiation condition, which forbids waves incoming from infinity, may be used to determine the contour of integration.

In the Appendix some aspects of the steady state solution are explained qualitatively by means of the replacement of the diffraction problem with boundaries by the equivalent problem of an infinite array of mirror sources on the y-axis, radiating in an unbounded medium.

In the next section we will give a mathematical justification of our steady state solution obtained in this section. We will solve the initial value problem (2.3), (2.4) and (2.5) and show that for $t\to\infty$ its solution tends uniquely to the steady state solution (4.6a,b).

5. The initial value problem

Consider now the complete initial value problem as given by eq. (2.3) with initial conditions (2.4) and boundary conditions (2.5). We will solve the initial value problem by applying first a

Fourier-transform with respect to the variable x and subsequently a Laplace-transform with respect to the time t.
Transforming the initial value problem we get:

$$\tilde{\Psi}_{yy}-\tilde{\gamma}^2(\lambda,s)\tilde{\Psi}=\frac{\delta(y)}{s-i},\quad \tilde{\gamma}^2(\lambda,s)=\beta^2\lambda^2-2ikM\lambda s+k^2s^2\,, \tag{5.1}$$

where

$$\tilde{\Psi}(\lambda,y,s)=\int_0^\infty e^{-st}\Psi(\lambda,y,t)dt=\int_0^\infty e^{-st}\left[\int_{-\infty}^\infty e^{i\lambda x}\psi(x,y,t)dx\right]dt\quad .$$

The boundary conditions for $\tilde{\Psi}$ are $\tilde{\Psi}_y(\lambda,\pm1,s)=0$. For real values of λ the function $\tilde{\gamma}(\lambda,s)$ has two branch points in the complex s-plane, viz. $s=i(M\pm1)\lambda k^{-1}$. We make $\tilde{\gamma}$ a single-valued function of s by cutting the s-plane along the imaginary axis between these two branch points and by requiring that $\tilde{\gamma}$ should behave like ks for $|s|\to\infty$.

The solution of eq. (5.1) satisfying the boundary conditions is given by:

$$\tilde{\Psi}(\lambda,y,s)=\frac{1}{s-i}\left[\frac{e^{\tilde{\gamma}(|y|-1)}+e^{-\tilde{\gamma}(|y|-1)}}{2\tilde{\gamma}(e^{-\tilde{\gamma}}-e^{\tilde{\gamma}})}\right].$$

The formal, unique solution of our initial value problem is given by:

$$\psi(x,y,t)=\frac{1}{4\pi^2 i}\int_\infty^\infty e^{-i\lambda x}\left[\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}e^{st}\tilde{\Psi}(\lambda,y,s)ds\right]d\lambda\,,$$

where the integration in the complex s-plane is performed along a contour to the right of the singularities of $\tilde{\Psi}$ in the s-plane and in the complex λ-plane along the real λ-axis.
The singularities of $\tilde{\Psi}$ only consist of simple poles which are seen to be $s=i$ and the infinite number of points $s_n^\pm$ satisfying the equation $\tilde{\gamma}^2(\lambda,s)=-n^2\pi^2$, $n=0,1,2,\ldots$ A calculation shows that

$$s_n^\pm=\frac{i}{k}\left[M\lambda\pm\sqrt{\lambda^2+n^2\pi^2}\right],\qquad n=0,1,2,\ldots\,.$$

Hence all singularities of $\tilde{\Psi}$ are situated on the imaginary s-axis. Calculus of residues then gives after some manipulation:

$$\psi(x,y,t)=\frac{1}{4\pi}\int_{-\infty}^\infty e^{-i\lambda x}e^{it}\left[\frac{e^{\gamma(|y|-1)}+e^{-\gamma(|y|-1)}}{\gamma(e^{-\gamma}-e^{\gamma})}+\right.$$
$$\left.+\sum_{n=0}^{\infty}{}'\frac{(-1)^{n+1}\cos\{n\pi(|y|-1)\}}{k\sqrt{\lambda^2+n^2\pi^2}}\left\{\frac{e^{if_n^-(\lambda)t}}{f_n^-(\lambda)}-\frac{e^{if_n^+(\lambda)t}}{f_n^+(\lambda)}\right\}\right]d\lambda\,, \tag{5.2}$$

where Σ' means that half the contribution for $n=0$ should be taken and where

$$f_n^\pm(\lambda)=\frac{1}{k}\left(M\lambda-k\pm\sqrt{\lambda^2+n^2\pi^2}\right),\qquad n=0,1,2,\ldots\,.$$

The first fraction in the integrand, representing the residue of $s=i$, is completely analogous to the Fourier-transform of the steady state solution and hence possesses again poles on the real λ-axis. This seems rather disastrous but we will show that these singularities are cancelled for finite time t by the first $N+1$ terms of the infinite series between square brackets.
Therefore we rewrite eq. (5.2) as:

$$\psi(x,y,t)=\frac{1}{4\pi}\int_{-\infty}^\infty e^{-i\lambda x}e^{it}\left[\frac{e^{\gamma(|y|-1)}+e^{-\gamma(|y|-1)}}{\gamma(e^{-\gamma}-e^{\gamma})}+\sum_{n=0}^{N}{}' r_n(\lambda,t)\ d\lambda\right]+$$
$$+\frac{1}{4\pi}\int_{-\infty}^\infty e^{-i\lambda x}e^{it}\sum_{n=N+1}^{\infty}r_n(\lambda,t)d\lambda\,, \tag{5.3}$$

where

$$r_n(\lambda, t) = \frac{(-1)^{n+1}\cos\{n\pi(|y|-1)\}}{k\sqrt{\lambda^2+n^2\pi^2}}\left\{\frac{e^{if_n^-(\lambda)t}}{f_n^-(\lambda)} - \frac{e^{if_n^+(\lambda)t}}{f_n^+(\lambda)}\right\}. \tag{5.4}$$

Making use of the identity

$$f_n^+(\lambda)f_n^-(\lambda) = -k^{-2}(\gamma^2+n^2\pi^2) = -\beta^2 k^{-2}(\lambda-\xi_n^-)(\lambda-\xi_n^+),$$

an alternative form of $r_n(\lambda, t)$ becomes:

$$r_n(\lambda, t) = \frac{(-1)^{n+1}\cos\{n\pi(|y|-1)\}}{\sqrt{\lambda^2+n^2\pi}}\cdot\frac{k}{\beta^2}\,\frac{f_n^-\,e^{if_n^+t}-f_n^+\,e^{if_n^-t}}{(\lambda-\xi_n^-)(\lambda-\xi_n^+)}.$$

It is seen now that the finite sum to $N+1$ terms in the first integral of eq. (5.3) has exactly the same $2N+2$ poles $\xi_n^\pm$, $n=0, 1, \ldots, N$, on the real λ-axis as the first item of this integrand. Hence by showing that these poles mutually cancel each other for finite time t it is clear that the integrand of the first integral of eq. (5.3) is an analytic function of λ in a small neighbourhood of the real λ-axis. Each term of the infinite series in the second integral in eq. (5.3) has simple poles in the complex λ-plane with non-zero imaginary parts. Making use of the sine-product ([9]):

$$\gamma(e^{-\gamma}-e^{\gamma}) = 2i\gamma\sin i\gamma = -2\gamma^2\prod_{n=1}^{\infty} n^{-2}\pi^{-2}(\gamma-n\pi i)(\gamma+n\pi i) =$$

$$= -2\gamma^2\prod_{n=1}^{\infty}\beta^2 n^{-2}\pi^{-2}(\lambda-\xi_n^-)(\lambda-\xi_n^+),$$

we may denote the expression between square brackets in eq. (5.3) as $F(\lambda)=N(\lambda)/D(\lambda)$, where:

$$D(\lambda) = \gamma^2\prod_{n=1}^{\infty}\beta^2 n^{-2}\pi^{-2}(\lambda-\xi_n^-)(\lambda-\xi_n^+),$$

$$N(\lambda) = -\tfrac{1}{2}[e^{-\gamma(|y|-1)}+e^{\gamma(|y|-1)}] +$$

$$-\tfrac{1}{2}k\lambda^{-1}(f_0^-\,e^{if_0^+t}-f_0^+\,e^{if_0^-t})\prod_{j=1}^{\infty}\beta^2 j^{-2}\pi^{-2}(\lambda-\xi_j^+)(\lambda-\xi_j^-) +$$

$$+\sum_{n=1}^{N}\frac{(-1)^{n+1}\cos\{n\pi(|y|-1)\}}{\sqrt{\lambda^2+n^2\pi^2}}\,k\frac{f_n^-\,e^{if_n^+t}-f_n^+e^{if_n^-t}}{n^2\pi^2}\,\gamma^2\prod_{\substack{j=1\\ j\neq n}}^{\infty}\beta^2 j^{-2}\pi^{-2}(\lambda-\xi_j^-)(\lambda-\xi_j^+).$$

We will show that $N(\xi_n^\pm)=0$, $n=0, 1, 2, \ldots N$. This means that $F(\lambda)=N(\lambda)/D(\lambda)$ has a zero residue for $\lambda=\xi_n^\pm$, $n=0, 1, \ldots, N$, and hence it is an analytic function of λ in a heighbourhood of these points. First we remark that the $\lambda=\xi_n^\pm$, $n=0, 1, \ldots N$, satisfy the equation $f_n^+(\lambda)=0$. This follows by substitution of the equality

$$(\xi_n^\pm)^2+n^2\pi^2 = (M\xi_n^\pm-k)^2$$

into the expression for $f_n^+(\lambda)$. So we have for $n=1, 2, \ldots, N$:

$$N(\xi_n^+) = -\cos\{n\pi(|y|-1)\}\left[1+\frac{(-1)^{n+1}kf_n^-(\xi_n^+)}{\sqrt{(\xi_n^+)^2+n^2\pi^2}}\prod_{\substack{j=1\\ j\neq n}}^{\infty}\left\{1-\left(\frac{n}{j}\right)^2\right\}\right] =$$

$$= -\cos\{n\pi(|y|-1)\}\left[1+\frac{kf_n^-(\xi_n^+)}{2\sqrt{(\xi_n^+)^2+n^2\pi^2}}\right] = -\cos\{n\pi(|y|-1)\}\frac{f_n^+(\xi_n^+)}{2\sqrt{(\xi_n^+)^2+n^2\pi^2}} = 0.$$

In a similar way one shows $N(\xi_n^-)=0$, $n=1, 2, \ldots N$ and $N(\xi_0^\pm)=0$.
So $F(\lambda)$ is an analytic function of λ in a small neighbourhood of the real λ-axis and hence by Cauchy's theorem the path of integration may be deformed within this neighbourhood. We will now deform this path of integration in such a way that the first integral of eq. (5.3) is split up into a number of contributions of which the behaviour for $t\to\infty$ is apparent. This is done by circumventing the $2N+2$ points $\xi_n^\pm$, $n=0,1, \ldots, N$ on the real λ-axis by small semi-circles around ξ_n^- in the upper halfplane and around ξ_n^+ in the lower halfplane (c.f. fig. 1). With this choice of the contour L all functions $f_n^+(\lambda)$ have positive imaginary part on the semi-circles. This may be concluded from

$$\operatorname{Im} f_n^+(\xi_n^\pm+i\varepsilon)=\operatorname{Im}\left[\frac{i\varepsilon(\beta^2\xi_n^\pm+Mk)}{k|M\xi_n^\pm-k|}\right]>0\,,$$

where $\operatorname{Re}(\varepsilon)>0$ for ξ_n^- and $\operatorname{Re}(\varepsilon)<0$ for ξ_n^+.
As the number of terms in the integrand of the first integral of eq. (5.3) is finite we are allowed to integrate term-by-term. With $r_n(\lambda, t)$ as stated in eq. (5.4) all terms containing the $f_n^-(\lambda)$ are each separately analytic functions of λ in a small neighbourhood of the real λ-axis. Hence we may transform back the contour L into the real λ-axis for these terms.

Taking now the limit $t\to\infty$ we first note that the integral over L of the finite sum, containing the $f_n^+(\lambda)$ only, tends to zero because the semi-circles are chosen in such a way that these contributions tend to zero exponentially in t and the remaining parts of the real λ-axis make contributions that die out like $1/t$: this may be shown easily by integration by parts. It is clear now that in accordance with this remark the integration over the real λ-axis of the finite sum containing the $f_n^-(\lambda)$ only, yields a contribution that also dies out like $1/t$. The inverse Fourier transform of the remaining infinite series of generalized functions may be taken term by term ([4]). By the method of stationary phase it may be shown that each term of the series behaves for large t like $n^{-\frac{1}{2}}t^{-\frac{1}{2}}$ from which we may conclude that the second integral in eq. (5.3) dies out like $t^{-\frac{1}{2}}$ for t tending to infinity.

Hence we have:

$$\lim_{t\to\infty}\psi(x, y, t)=\frac{e^{it}}{4\pi}\int_L e^{-i\lambda x}\left[\frac{e^{\gamma(|y|-1)}+e^{-\gamma(|y|-1)}}{\gamma(e^{-\gamma}-e^{\gamma})}\right]d\lambda\,.$$

This shows that the original steady state solution as derived in section 4 is the correct solution of the initial value problem in the limit $t\to\infty$.

6. The axisymmetric case

In this section we will give some results for the three-dimensional case of an acoustic source radiating in a cylindrical tube. Introducing dimensionless cylindrical coordinates (x, r, φ) we have a source at $x=0$, $r=0$ and a perfectly reflecting cylindrical tube given by $r=1$. The complete axisymmetric initial value problem then becomes (Le Grand [3]):

$$\beta^2\psi_{xx}+\frac{1}{r}(r\psi_r)_r-2kM\psi_{xt}-k^2\psi_{tt}=\frac{1}{r}\delta(r)\delta(x)e^{it},\quad t>0\,. \tag{6.1}$$

with initial conditions: $\psi(x, r, 0)=\psi_t(x, r, 0)=0$, and boundary conditions $\psi_r(x, 1, t)=0$, and $\psi_r(x, 0, t)=0$, $(x\neq 0)$. Quite analogous to the two-dimensional case a steady state solution may be derived for this problem, which may be made unique by introducing a small bulk viscosity. Note, however, that in this case where also incoming waves generated at infinity will occur, no use can be made of the mirror source principle to give a qualitative explanation of that phenomenon.

By the same argument as in the two-dimensional case we may solve the initial value problem by first applying a Fourier transform with respect to x and then a Laplace transform with respect to time t. This yields the following boundary value problem for the Laplace-Fourier transform $\tilde{\Psi}(\lambda, r, s)$ of $\psi(x, r, t)$:

$$\tilde{\Psi}_{rr} + \frac{1}{r}\tilde{\Psi}_r - \tilde{\gamma}^2(\lambda, s)\tilde{\Psi} = \frac{\delta(r)}{r(s-i)}, \quad \tilde{\Psi}_r(\lambda, 1, s) = \tilde{\Psi}_r(\lambda, 0, s) = 0\,, \tag{6.2}$$

where $\tilde{\gamma}(\lambda, s)$ is the same as in section 5. The solution of this problem is given by

$$\tilde{\Psi}(\lambda, r, s) = \frac{-1}{2(s-i)}[I_1(\tilde{\gamma})]^{-1}[K_1(\tilde{\gamma})I_0(\tilde{\gamma}r) + I_1(\tilde{\gamma})K_0(\tilde{\gamma}r)]\,, \tag{6.3}$$

where the I_0, I_1, K_0 and K_1 denote the modified Bessel functions ([9]). The formal solution of our initial value problem is then given by

$$\psi(x, r, t) = \frac{1}{4\pi^2 i}\int_{-\infty}^{\infty} e^{-i\lambda x}\left[\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^{st}\tilde{\Psi}(\lambda, r, s)ds\right]d\lambda\,, \tag{6.4}$$

where the integrations in the s- and λ-planes are performed as in section 5. The integral with respect to s may be evaluated by calculus of residues. This gives after some manipulation:

$$\psi(x, r, t) = \frac{-1}{4\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}e^{it}\left[\frac{K_1(\gamma)I_0(\gamma r) + I_1(\gamma)K_0(\gamma r)}{I_1(\gamma)} + \sum_{j=0}^{\infty} r_j(\lambda, t)\right]d\lambda\,, \tag{6.5}$$

with

$$r_j(\lambda, t) = \frac{\tilde{\gamma}(\lambda, s_j^-)K_1\{\tilde{\gamma}(\lambda, s_j^-)\}I_0\{r\tilde{\gamma}(\lambda, s_j^-)\}}{I_0\{\tilde{\gamma}(\lambda, s_j^-)\}k^2\mu_j}\left\{\frac{e^{if_j^- t}}{f_j^-} - \frac{e^{if_j^+ t}}{f_j^+}\right\}, \tag{6.6}$$

where $s_j^\pm = iM\lambda k^{-1} \pm i\mu_j$, $j = 0, 1, \ldots$ are the zeros of $I_1\{\tilde{\gamma}(\lambda, s)\}$ in the complex s-plane. Making use of the identity

$$f_j^+(\lambda)f_j^-(\lambda) = -\beta^2 k^{-2}(\lambda - \xi_j^-)(\lambda - \xi_j^+)\,,$$

where the $\xi_j^\pm$ are the zeros of $I_1\{\gamma(\lambda)\}$ in the complex λ-plane, the function $r_j(\lambda, t)$ may be written as

$$r_j(\lambda, t) = \frac{\tilde{\gamma}(\lambda, s_j^-)K_1\{\tilde{\gamma}(\lambda, s_j^-)\}I_0\{r\tilde{\gamma}(\lambda, s_j^-)\}}{I_0\{\tilde{\gamma}(\lambda, s_j^-\}\mu_j\beta^2}\cdot\frac{f_j^- e^{if_j^+ t} - f_j^+ e^{if_j^- t}}{(\lambda - \xi_j^-)(\lambda - \xi_j^+)}\,.$$

With the aid of the product development of $I_1(\gamma)$, ([9]),

$$I_1(\gamma) = \tfrac{1}{2}\gamma \prod_{j=1}^{\infty} \beta^2\gamma^{-2}(\xi_j^\pm)(\lambda - \xi_j^-)(\lambda - \xi_j^+)\,,$$

it may be shown that all real singularities of the first item between square brackets in eq. (6.5) and of the infinite series, all occurring in the first $N+1$ terms only, mutually cancel each other for finite time t. The complete integrand will therefore be divided into two portions, one containing the first item between square brackets and terms $r_n(\lambda, t)$ which have singularities on the path of integration and one containing all others. Because of the analyticity of the first portion in a small neighbourhood of the real λ-axis we may again deform the path of integration into L by Cauchy's theorem (c.f. fig. 1).

The reason for choosing L in this way is exactly the same as in section 5, so is the limitprocess $t \to \infty$. Hence we have

$$\lim_{t\to\infty} \psi(x, y, t) = -\frac{e^{it}}{4\pi}\int_L e^{-i\lambda x}\left[\frac{K_1(\gamma)I_0(\gamma r) + I_1(\gamma)K_0(\gamma r)}{I_1(\gamma)}\right]d\lambda\,.$$

This represents the Fourier transform of the steady state solution in the limit $t \to \infty$.

Appendix

The free source solution and the mirror sources principle.

The initial value problem for a single two-dimensional source, situated at $x=0$, $y=0$ and switched on at time $t=0$, in an unbounded homogeneous medium is given by eq. (2.3) together with the initial conditions $\psi(x, y, 0)=\psi_t(x, y, 0)=0$. Let us consider this initial value problem in a $\tilde{x}$, $\tilde{y}$, $\tilde{t}$-coordinate system moving with the fluid:

$$\tilde{x}=x-Mk^{-1}\tilde{t}\,,\quad \tilde{y}=y\,,\quad \tilde{t}=k^{-1}t\,.$$

Then eq. (2.3) becomes:

$$\psi_{\tilde{x}\tilde{x}}+\psi_{\tilde{y}\tilde{y}}-\psi_{\tilde{t}\tilde{t}}=\delta(\tilde{x}+M\tilde{t})\delta(\tilde{y})e^{ik\tilde{t}}\,,\quad \tilde{t}>0$$

with initial conditions $\psi(\tilde{x}, \tilde{y}, 0)=\psi_t(\tilde{x}, \tilde{y}, 0)=0$. The solution of this initial value problem is wellknown and may be found in [10]. It is given by

$$\psi(\tilde{x}, \tilde{y}, \tilde{t})=-\frac{1}{2\pi}\int_0^{\tilde{t}} d\tau \iint \frac{e^{ik\tau}\,\delta(\xi+M\tau)\,\delta(\eta)\,d\xi\,d\eta}{[(\tilde{t}-\tau)^2-(\tilde{x}-\xi)^2-(\tilde{y}-\eta)^2]^{\frac{1}{2}}}\,,$$

where the domain of integration in the ξ, η-plane consists of all points satisfying the inequality

$$(\tilde{x}-\xi)^2+(\tilde{y}-\eta)^2\leqq(\tilde{t}-\tau)^2\,.$$

Performing the integration in the ξ, η-plane and returning to the original x, y, t-variables, we find

$$\psi(x, y, t)=-\frac{1}{2\pi}\int_0^{t/k}\frac{e^{ik\tau}\,d\tau}{[(\tau-t/k)^2+(x+M\tau-Mt/k)^2-y^2]^{\frac{1}{2}}}\,,$$

if the square root is real and

$$\psi(x, y, t)=0\,,$$

if the square root is imaginary.

After substitution of $\theta=t/k-\tau$ we get after some manipulation

$$\psi(x, y, t)=\begin{cases} 0\,, & \text{for}\quad t/k<\theta_0=\beta^{-2}[-Mx+\sqrt{x^2+\beta^2y^2}] \\ -\dfrac{e^{it}}{2\pi}\displaystyle\int_{\theta_0}^{t/k}\frac{e^{-ik\theta}\,d\theta}{[\theta^2-(x-M\theta)^2-y^2]^{\frac{1}{2}}}\,, & \text{for}\quad t/k\geqq\theta_0 \end{cases}$$

where θ_0 is the positive root of the equation

$$\theta^2-(x-M\theta)^2-y^2=0\,.$$

Restricting our attention to the case $t/k\geqq\theta_0$, the integral may be transformed into

$$\psi(x, y, t)=-\frac{e^{it}}{2\pi\beta}e^{ikM\beta^{-2}x}\int_1^{a(t)}\frac{\exp[-ik\beta^{-2}\tau\sqrt{x^2+\beta^2y^2}]}{\sqrt{\tau^2-1}}\,d\tau\,,$$

where the upper bound of integration $a(t)$ tends to infinity for $t\to\infty$. In the limit $t\to\infty$ this upper bound may be replaced by ∞ and the integral is expressed in terms of the Hankel-function $H_0^{(2)}$:

$$\psi(x, y, t)=\tfrac{1}{4}i\beta^{-1}\exp(ikM\beta^{-2}x)H_0^{(2)}\left\{k\beta^{-2}\sqrt{x^2+\beta^2y^2}\right\}e^{it}\,,$$

showing that for a single source in a homogeneous unbounded medium the steady state is reached for $t\to\infty$.

Our problem of a single steady source situated at $x=0$, $y=0$ radiating between two parallel, perfectly reflecting walls $y=\pm 1$ may be replaced by the equivalent problem of the radiation of an infinite array of mirror sources of equal strength and phase, situated at $x=0$, $y=2j$, $j=0, \pm 1, \pm 2, \ldots$ in a medium without boundaries ([10]). This means that an alternative form of the steady state solution is given by

$$\psi(x, y, t) = \tfrac{1}{4} i\beta^{-1} \sum_{j=-\infty}^{\infty} \exp(ikM\beta^{-2}x) H_0^{(2)}\{k\beta^{-2}\sqrt{x^2+\beta^2(y-2j)^2}\}\, e^{it}$$

For large values of $x^2+\beta^2 y^2$ we may use the asymptotic representation of the Hankel functions ([9]) to give

$$\psi(x, y, t) \sim \sum_{j=-\infty}^{\infty} \left\{\frac{2\beta^2}{\pi k\sqrt{x^2+\beta^2(y-2j)^2}}\right\}^{\frac{1}{2}} \exp\left[ik\beta^{-2}(Mx-\sqrt{x^2+\beta^2(y-2j)^2})\right] e^{it}$$

From this equation it can be seen that the approximate form of the equi-phase curves (or wavefronts) for fixed t of the mirror source situated at $x=0$, $y=2j$ is given by

$$Mx-\sqrt{x^2+\beta^2(y-2j)^2} = -c$$

where c is a positive constant. This equation can be rewritten as

$$(x-Mc\beta^{-2})^2+(y-2j)^2 = c^2\beta^{-2}$$

Hence the wavefronts for fixed t are approximately circles with center $(Mc\beta^{-2}, 2j)$ and radius c/β. Fixing our attention to a fixed point (x^*, y^*) with $x^*>0$ in the region between the walls $y=\pm 1$, it is clear now, that for all sources sufficiently far away from the origin, the center of the circular wavefront running through (x^*, y^*) lies to the right of the vertical $x=x^*$ and hence the outward normal to this wavefront at x^*, y^* has a negative x-component, representing a local phase-velocity in the direction of the negative x-axis, in other words an incoming wave. Only a finite number of mirror sources gives rise to a positive x-component of the phase-velocity at x^*, y^*. For a point x^*, y^*, with $x^*<0$ only phase-velocities in the negative x-direction are possible, all representing outgoing waves.

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